LARGE SETS OF COMPLEX AND REAL EQUIANGULAR LINES

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Abstract. Large sets of equiangular lines are constructed from sets of mutually unbiased bases, over both the complex and the real numbers.

1. Introduction

The *angle* between vectors $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$ of unit norm in $\mathbb{C}^d$ is $\arccos |\langle \boldsymbol{x}_j, \boldsymbol{x}_k \rangle|$, where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product. A set of m distinct lines in $\mathbb{C}^d$ through the origin, represented by vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ of equal norm, is *equiangular* if for some real constant a we have

$$|\langle \boldsymbol{x}_j, \boldsymbol{x}_k \rangle| = a \quad \text{for all } j \neq k.$$

The number of equiangular lines in $\mathbb{C}^d$ is at most d^2 [4], and when the vectors are further constrained to lie in $\mathbb{R}^d$ this number is at most $d(d+1)/2$ (attributed to Gerzon in [9]). It is an open question, in both the complex and real case, whether the upper bound can be attained for infinitely many d, although in both cases $\Theta(d^2)$ equiangular lines exist for all d. Specifically, König [8] constructed $d^2 - d + 1$ equiangular lines in $\mathbb{C}^d$ where $d - 1$ is a prime power, and de Caen [3] constructed $2(d+1)^2/9$ equiangular lines in $\mathbb{R}^d$ where $(d+1)/3$ is twice a power of 4. By extending vectors using zero entries as necessary, we can derive sets of $\Theta(d^2)$ equiangular lines from these direct constructions for all d.

Two orthogonal bases $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d\}, \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_d\}$ for $\mathbb{C}^d$ are *unbiased* if

$$\frac{|\langle \boldsymbol{x}_j, \boldsymbol{y}_k \rangle|}{||\boldsymbol{x}_j|| \cdot ||\boldsymbol{y}_k||} = \frac{1}{\sqrt{d}} \quad \text{for all } j, k. \tag{1}$$

A set of orthogonal bases is a set of *mutually unbiased bases* (MUBs) if all pairs of distinct bases are unbiased.

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The number of MUBs in $\mathbb{C}^d$ is at most $d+1$ [4, Table I], which can be attained when d is a prime power by a variety of methods [5], [7], [10]. The number of MUBs in $\mathbb{R}^d$ is at most $d/2+1$ [4, Table I], which can be attained when d is a power of 4 [1], [2].

The authors recently gave a direct construction of $d^2/4$ equiangular lines in $\mathbb{C}^d$, where $d/2$ is a prime power [6]. We show here how to generalize the underlying construction to give $\Theta(d^2)$ equiangular lines in $\mathbb{C}^d$ and $\mathbb{R}^d$ directly from sets of complex and real MUBs.

2. The Construction

We associate an ordered set of m vectors in $\mathbb{C}^d$ with the $m \times d$ matrix formed from the vector entries, using the ordering of the set to determine the ordering of the vectors.

Theorem 1. *Suppose that $B_1, B_2, \ldots, B_r$ form a set of r MUBs in $\mathbb{C}^d$, each of whose vectors has all entries of unit magnitude, where $r \le d$. Let $a_1, a_2, \ldots, a_t$ be constants in $\mathbb{C}$, where $t \ge 1$. Let $B_j(v)$ be the set of d vectors formed by multiplying entry j of each vector of B_j by $v \in \mathbb{C}$, and let $L(v) = \cup_{j=1}^r B_j(v)$ (considered as an ordered set). Then all inner products between distinct vectors among the rd vectors of*

$$\Big[L(a_1) \quad L(a_2) \quad \ldots \quad L(a_t) \quad L\Big(t+1-\sum_{j=1}^t a_j\Big)\Big]$$

in $\mathbb{C}^{(t+1)d}$ have magnitude $\sum_{j=1}^t |a_j-1|^2 + \Big|\sum_{j=1}^t (a_j-1)\Big|^2$ or $(t+1)\sqrt{d}$.

Proof. Write $A = \{a_1, a_2, \ldots, a_t, t+1-\sum_{j=1}^t a_j\}$ for the set of arguments $v \in \mathbb{C}$ taken by $L(v)$ in the construction. We consider two cases, according to whether distinct vectors of $L(v)$ originate from the same basis or from distinct bases.

In the first case, consider the inner product of distinct vectors of $L(v)$ constructed from vectors from the same basis B_j. Since the original vectors are orthogonal, this inner product is $z(|v|^2-1)$ for some z of unit magnitude that depends only on the original two vectors. Since each occurrence of $L(v)$ uses the same ordering, the inner product of the corresponding concatenated vectors in $\mathbb{C}^{(t+1)d}$ is therefore $z\sum_{v\in A}(|v|^2-1)$, which equals $z\Big(\sum_{j=1}^t |a_j-1|^2 + \Big|\sum_{j=1}^t (a_j-1)\Big|^2\Big)$ after straightforward algebraic manipulation.

In the second case, consider vectors of $L(v)$ constructed from vectors from distinct bases B_j, B_k . Let these constructed vectors be

$$\begin{aligned} \boldsymbol{x} &= (x_1 \quad x_2 \quad \ldots \quad vx_j \quad \ldots \quad \ldots \quad x_d), \\ \boldsymbol{y} &= (y_1 \quad y_2 \quad \ldots \quad \ldots \quad vy_k \quad \ldots \quad y_d). \end{aligned}$$

The inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$ in $L(v)$ is

$$x_1\overline{y_1}+\cdots+vx_j\overline{y_j}+\cdots+\overline{v}x_k\overline{y_k}+\cdots+x_d\overline{y_d}=\sum_{\ell=1}^{d}x_\ell\overline{y_\ell}+(v-1)x_j\overline{y_j}+(\overline{v}-1)x_k\overline{y_k}.$$

Therefore the corresponding concatenated vectors in $\mathbb{C}^{(t+1)d}$ have inner product

$$(t+1)\sum_{\ell=1}^{d}x_\ell\overline{y_\ell}+x_j\overline{y_j}\sum_{v\in A}(v-1)+x_k\overline{y_k}\sum_{v\in A}(\overline{v}-1)=(t+1)\sum_{\ell=1}^{d}x_\ell\overline{y_\ell},$$

because $\sum_{v\in A}v=t+1$. Now, all of the entries x_ℓ, y_ℓ have unit magnitude by assumption, and so $\left|\sum_{\ell=1}^{d}x_\ell\overline{y_\ell}\right|=\sqrt{d}$ by the MUB property (1). Therefore the concatenated vectors in $\mathbb{C}^{(t+1)d}$ have inner product of magnitude $(t+1)\sqrt{d}$. $\square$

Remark. Lemma 6.2 of [6] describes the special case $t=1$ and $r=d$ of Theorem 1, in which the MUBs are constrained to arise from a $(d,d,d,1)$ relative difference set in an abelian group according to the construction method of [5]; the permutation π given in [6, Lemma 6.2] can be dropped without loss of generality.

Corollary 2. *Let t be a positive integer and let d be a prime power. There exist d^2 equiangular lines in $\mathbb{C}^{(t+1)d}$.*

Proof. There exists a set of $d+1$ MUBs in $\mathbb{C}^d$ for which one of the bases is the standard basis [10]. After appropriate scaling, all entries of each of the vectors of the remaining d bases therefore have unit magnitude, using (1). So we may apply Theorem 1 with $r=d$.

There are infinitely many choices of $a_1,a_2,\ldots,a_t\in\mathbb{C}$ for which the two magnitudes in the conclusion of Theorem 1 are equal, one such choice being $a_j=1+d^{1/4}/\sqrt{t}$ for each j. $\square$

Corollary 3. *Let t be a positive integer and let d be a power of 4. There exist $d^2/2$ equiangular lines in $\mathbb{R}^{(t+1)d}$.*

Proof. There exists a set of $d/2+1$ MUBs in $\mathbb{R}^d$ for which one of the bases is the standard basis [1], [2]. Apply Theorem 1 with $r=d/2$ and take, for example, $a_j=1+d^{1/4}/\sqrt{t}$ for each j to obtain real equiangular lines. $\square$

The proof of Theorem 1 shows that the magnitude of the inner product of distinct vectors is $\sum_{v\in A}(|v|^2-1)$ or $(t+1)\sqrt{d}$. In the construction of Corollaries 2 and 3, the constants a_j are chosen so that these magnitudes are equal, and the inner product of each concatenated vector with itself is $\sum_{v\in A}(|v|^2+d-1)$. It follows that the common angle for the sets of equiangular lines constructed in Corollaries 2 and 3 is $\arccos(1/(1+\sqrt{d}))$ for all t, regardless of the choice of the constants a_j.

Theorem 1 can be generalized as follows. Let $c_1, \dots, c_t$ be real constants, and take the rd vectors of

$$\Big[c_1 L(a_1) \quad c_2 L(a_2) \quad \dots \quad c_t L(a_t) \quad L\Big(1+\sum_{j=1}^{t} c_j^2(1-a_j)\Big)\Big]$$

in $\mathbb{C}^{(t+1)d}$. Then all inner products between distinct vectors have magnitude $\sum_{j=1}^t c_j^2 \left|1-a_j\right|^2 + \left|\sum_{j=1}^t c_j^2(1-a_j)\right|^2$ or $(1+\sum_{j=1}^t c_j^2)\sqrt{d}$. If $a_1, a_2, \dots, a_t$ and $c_1, c_2, \dots, c_t$ are chosen so that these two magnitudes are equal, the common angle of the resulting set of equiangular lines is again $\arccos(1/(1+\sqrt{d}))$.

Remark. The case $t = 1$ and $d = 4$ of Corollary 3 constructs 8 equiangular lines in $\mathbb{R}^8$ having the form $[L(a) \quad L(2-a)]$, where $a \in \{1 \pm \sqrt{2}\}$. We can extend this to a set $\left[\begin{smallmatrix} L(a) & L(2-a) \\ L(2-a) & L(a)\end{smallmatrix}\right]$ of 16 equiangular lines in $\mathbb{R}^8$, where $a \in \{1 \pm \sqrt{2}\}$; this extension does not seem to generalize easily to larger values of d.

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